

# Characterization of the Structure-Generating Functions of Regular Sets and the DOL Growth Functions

TAKUYA KATAYAMA,\* MASAYUKI OKAMOTO,<sup>†</sup> AND HAJIME ENOMOTO\*

\* *Department of Computer Science, Tokyo Institute of Technology, Tokyo, Japan, and*

<sup>†</sup> *Department of Computer Science, Shinshu University, Nagano, Japan*

The structure-generating functions of regular sets and the DOL growth functions are characterized. Our result is: A rational function  $f(z)$  with integral coefficients is a structure-generating function of a regular set if and only if (1) the constant term of its denominator is 1 and that of its numerator is 0, (2) every coefficient  $a_n$  of its Taylor series expansion is nonnegative, and (3) every pole of the minimal absolute value of  $f_i(z) = \sum_{n=0}^{\infty} a_{nM+i} z^n$  is of the form  $r\epsilon$ , where  $r > 0$  and  $\epsilon$  is a root of unity for any integer  $M \geq 1$  and  $i = 0, 1, \dots, M-1$ . Also stated are a result on the star height problem and an analogous characterization of the growth functions of DOL systems.

## 1. INTRODUCTION

The structure-generating functions (sgf's) of  $\epsilon$ -free regular sets are characterized.

Kuich (1970), in his study of the entropy of context-free languages, has shown that the sgf's are rational functions whose coefficients of the Taylor expansion are nonnegative integers. Berstel (1971) has shown that poles of the  $R_+$ -recognizable functions on the circle of convergence must be roots of unity, and from his result we know that the converse of Kuich's result does not hold.

In this paper we show that the inverse of a slightly modified version of Berstel's theorem holds and we characterize the structure-generating functions of regular sets. Our proof is, of course, constructive and we also show that every regular set has its sgf-equivalent counterpart whose star height is at most 2.

Our method is also applicable to DOL systems and a characterization of their growth functions is also stated.

The main results of this paper are obtained independently by Soittola (1976a,b) by different methods.

## 2. PRELIMINARIES

*Notation.* Throughout the paper we fix an infinite set  $\Sigma$  of letters and assume that every regular set  $R$  is constructed from a finite number of elements

of  $\Sigma$ , that is,  $R \subset \Sigma_1^*$  for some  $\Sigma_1 \subset \Sigma$ ,  $|\Sigma_1| < \infty$ . Without loss of generality we assume that regular sets are  $\epsilon$ -free and denote that class of all  $\epsilon$ -free regular sets by  $\mathcal{R}$ .

DEFINITION 1. A structure-generating function  $g_R$  of a regular set  $R$  is

$$g_R(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where  $a_n$  is the number of words of length  $n$  in  $R$ . It is known that  $g_R(z)$  can be expressed by a rational function of integral coefficients (Kuich, 1970) and we denote a class of such rational functions by  $\mathcal{G}$ ,

$$\mathcal{G} = \{g_R(z) \mid R \in \mathcal{R}\}.$$

DEFINITION 2.  $\mathcal{F}$  is a minimal class of rational functions which satisfies

- (1)  $z \in \mathcal{F}$ , and
- (2) if  $f, g \in \mathcal{F}$  then  $f + g$ ,  $f \cdot g$ ,  $f/(1 - f) \in \mathcal{F}$ , where  $z$  is a variable symbol and we call  $f/(1 - f)$  the *pseudo-inverse* of  $f$ .

DEFINITION 3.  $\mathcal{P}$  is a class of rational functions of the form

$$f(z) = \frac{q_1 z + \cdots + q_m z^m}{1 + p_1 z + \cdots + p_l z^l} = \sum_{n=1}^{\infty} a_n z^n,$$

where the  $p_i$ 's and  $q_i$ 's are integers and satisfy the following conditions.

- (1) Coefficients  $a_n$  of its Taylor expansion are nonnegative integers, and
- (2) every pole  $z_0$  of the minimal absolute value of  $f_i(z) = \sum_{n=0}^{\infty} a_{nM+i} z^n$  is of the form  $z_0 = r\epsilon$ , where  $r = |z_0|$  and  $\epsilon$  is a root of unity for any integer  $M \geq 1$  and  $i = 0, 1, \dots, M - 1$ .

DEFINITION 4. A partial Taylor series of order  $N$  of a rational function  $f(z) = Q(z)/P(z)$  is defined by

$$f(z) = \sum_{n=1}^N a_n z^n + \frac{Q_N(z)}{P(z)} z^{N+1}, \quad (2.1)$$

where  $P(z)$ ,  $Q(z)$ , and  $Q_N(z)$  are polynomials. This partial series can be easily obtained by dividing  $Q(z)$  by  $P(z)$ , starting from the lowest-order term of  $z$  and stopping after  $a_N$  has been obtained. Of course, the remainder  $[Q_N(z)/P(z)]z^{N+1}$  produces terms of higher order than  $N$ .

3. CHARACTERIZATION OF  $\text{sgf}$ 's OF REGULAR SETS

In this section we obtain a characterization theorem maintaining that  $\mathcal{G} = \mathcal{P}$  by showing that  $\mathcal{G} = \mathcal{F}$  and  $\mathcal{F} = \mathcal{P}$ . The first equality  $\mathcal{G} = \mathcal{F}$  is rather straightforward and  $\mathcal{F} \subset \mathcal{P}$  is due to Berstel; this section is mainly devoted to establishing  $\mathcal{F} \supset \mathcal{P}$ .

THEOREM 1.  $\mathcal{G} = \mathcal{F}$ .

*Proof.*  $\mathcal{G} \subset \mathcal{F}$  is immediate from the definition of  $\mathcal{F}$  and the fact that every regular set on  $\Sigma$  is obtained unambiguously by applying union, catenation, and catenation closure to finite subsets of  $\Sigma$  (Eilenberg, 1974, pp. 186).

$\mathcal{G} \supset \mathcal{F}$  is shown as follows by using  $|\Sigma| = \infty$ . Let  $f_1, f_2 \in \mathcal{G}$  and  $f_1 = g_{R_1}$ ,  $f_2 = g_{R_2}$ . Here we can choose  $R_1$  and  $R_2$  as  $R_1 \subset \Sigma_1^*$ ,  $R_2 \subset \Sigma_2^*$ , and  $\Sigma_1 \cap \Sigma_2 = \emptyset$ , and  $f_1 + f_2 = g_{R_1 \cup R_2} \in \mathcal{G}$ ,  $f_1 f_2 = g_{R_1 R_2} \in \mathcal{G}$ . As  $R_1$  is  $\epsilon$ -free,  $R_1 = a_1 S_1 \cup \dots \cup a_n S_n$ , where  $a_i \in \Sigma$ ,  $S_i = \{w \mid a_i w \in R_1\}$ . By introducing new symbols  $b_i \in \Sigma - \Sigma_1$ , where  $R_1 \subset \Sigma_1^*$ , and constructing  $R_0 = b_1 S_1 \cup \dots \cup b_n S_n$ , we can show that  $f_1/(1 - f_1) = g_{R_0^+} \in \mathcal{G}$ . Of course  $z = g_{\{a\}} \in \mathcal{G}$  ( $a \in \Sigma$ ) and this completes the proof.

The following lemma is used frequently without reference.

LEMMA 1. (i) Any polynomial with nonnegative integral coefficients belongs to  $\mathcal{F}$ . (ii) If  $f, g \in \mathcal{F}$  then  $g/(1 - f) = g + gf/(1 - f) \in \mathcal{F}$ .

LEMMA 2. If  $f(z) = \sum_{i=1}^k b_i z^i / (1 - cz) \in \mathcal{P}$ , that is,  $\sum_{i=1}^k b_i c^{k-i} \geq 0$  for  $1 \leq k \leq m$  and  $c > 0$ ,  $b_i$  are integers, then  $f(z) \in \mathcal{F}$ .

*Proof.* Straightforward by expanding  $f$  into a partial Taylor series.

In the following we fix an arbitrary function  $f(z)$  in  $\mathcal{P}$  and assume its form as

$$f(z) = \frac{Q(z)}{P(z)} = \frac{Q(z)}{(1 - \alpha_1 z)^{K_1} (1 - \alpha_2 z)^{K_2} \dots (1 - \alpha_s z)^{K_s}}, \quad (3.1)$$

where  $\alpha_1, \dots, \alpha_s$  are inverses of poles of  $f(z)$  and  $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_s|$ , and  $K_i$  is a nonnegative integer called "multiplicity of  $\alpha_i$ ." We will use "inverse pole" to denote the  $\alpha_i$ 's. As the expansion coefficients of  $f(z)$  are nonnegative we can easily show that

$$\alpha_1 = 1/r > 0,$$

where  $r$  is a radius of convergence of the Taylor series and therefore  $\alpha_1 > 0$  (Pringsheim's theorem). Without loss of generality we assume that  $Q(1/\alpha_1) \neq 0$  in what follows.

LEMMA 3. Assume that the multiplicity  $K_1$  of the inverse pole  $\alpha_1$  is 1 and  $\alpha_1 > |\alpha_2|, \dots, |\alpha_s|$  in (3.1). If  $f(z)$  is expanded into a partial Taylor series as in (2.1), where

$$Q_N(z) = q_N^0 + q_N^1 z + \dots + q_N^{m(N)} z^{m(N)},$$

then putting

$$P_1(z) = (1 - \alpha_2 z)^{K_2} \dots (1 - \alpha_s z)^{K_s}$$

we have

$$Q_N(z)/q_N^0 \rightarrow P_1(z) \quad \text{as } N \rightarrow \infty;$$

that is,

$$\lim_{N \rightarrow \infty} [q_N^i / q_N^0] = \text{coefficient of } z^i \text{ in } P_1(z).$$

*Proof.* Let  $P(z) = (1 - \alpha_1 z) P_1(z) = 1 + p_1 z + \dots + p_l z^l$ . From the definition of  $Q_N(z)$

$$Q_N(z) = q_N^0 P(z) + Q_{N+1}(z)z$$

and  $m(N) = l - 1$  for sufficiently large  $N$ , and we have

$$q_{N+1}^i = q_N^{i+1} - p_{i+1} q_N^0 \quad (i = 0, \dots, l-2),$$

$$q_{N+1}^{l-1} = -p_l q_N^0.$$

From these equalities, we have

$$q_N^0 + p_1 q_{N-1}^0 + \dots + p_l q_{N-l}^0 = 0 \quad (3.2)$$

and

$$q_N^i = q_{N+i}^0 + p_1 q_{N+i-1}^0 + \dots + p_i q_N^0.$$

As the characteristic equation of (3.2) is

$$x^l + p_1 x^{l-1} + \dots + p_l = (x - \alpha_1)(x - \alpha_2)^{K_2} \dots (x - \alpha_s)^{K_s} = 0,$$

so

$$q_N^0 = h \alpha_1^N + \sum_{i=2}^s \sum_{j=0}^{K_i-1} h_{ij} N^j \alpha_i^N$$

for some  $h, h_{ij}$ , and as  $\alpha_1 > |\alpha_k|$  ( $k = 2, \dots, s$ ) and  $h \neq 0$  by  $Q(1/\alpha_1) \neq 0$ ,

$$\lim_{N \rightarrow \infty} [q_N^i / q_N^0] = \alpha_1^i + p_1 \alpha_1^{i-1} + \dots + p_i.$$

On the other hand,  $P_1(z) = (1 + p_1z + \cdots + p_lz^l)/(1 - \alpha_1z)$  and the coefficient of  $z^i$  of  $P_1(z)$  is given by

$$\alpha_1^i + p_1\alpha_1^{i-1} + \cdots + p_i.$$

This completes the proof.

In the following we prove that any  $f \in \mathcal{P}$  is in  $\mathcal{F}$  step by step according to the condition that  $f$  satisfies.

**LEMMA 4.** *Let  $f \in \mathcal{P}$ . If the multiplicity  $K_1$  of the inverse pole  $\alpha_1$  is 1 and  $\alpha_1 \gg |\alpha_2|$  (explicitly  $\alpha_1 > 2(2^{1/m} + 1 + |\alpha_2|)/(2^{1/m} - 1)$ ,  $m = K_2 + \cdots + K_s$ ), then  $f \in \mathcal{F}$ .*

*Proof.* For some sufficiently large  $N$ , we expand  $f$  into a partial Taylor series as in (2.1). As  $a_n \geq 0$  ( $1 \leq n \leq N$ ),  $\sum_{n=1}^N a_n z^n \in \mathcal{F}$  and we need only prove that  $Q_N(z)z^{N+1}/P(z) \in \mathcal{F}$ .

Let  $d_1, \dots, d_m$  be

$$\begin{aligned} P_1(z) &= (1 - \alpha_2 z)^{K_2} \cdots (1 - \alpha_s z)^{K_s} \\ &= 1 + d_1 z + \cdots + d_m z^m \end{aligned}$$

and choose  $t = [\alpha_1/2]$ , where  $[x]$  means the largest integer which is not beyond  $x$ , and express  $P(z)$  as

$$\begin{aligned} P(z) &= (1 - \alpha_1 z)(1 + d_1 z + \cdots + d_m z^m) \\ &= 1 - tz - \{(\alpha_1 - d_1 - t)z + (\alpha_1 d_1 - d_2)z^2 + \cdots \\ &\quad + (\alpha_1 d_{m-1} - d_m)z^m + \alpha_1 d_m z^{m+1}\} \\ &= 1 - tz - (e_1 z + e_2 z^2 + \cdots + e_{m+1} z^{m+1}), \end{aligned}$$

where the  $e_i$ 's are integers given by

$$e_1 = \alpha_1 - d_1 - t, \quad e_{m+1} = \alpha_1 d_m,$$

and

$$e_i = \alpha_1 d_{i-1} - d_i \quad (i = 2, \dots, m).$$

Then

$$\frac{Q_N(z)z^{N+1}}{P(z)} = \frac{Q_N(z)z^{N+1}}{1 - tz} \left/ \left( 1 - \frac{\sum_{i=1}^{m+1} e_i z^i}{1 - tz} \right) \right.$$

Let  $E_k = \sum_{i=1}^k e_i t^{k-i}$  for  $k = 1, \dots, m+1$ . Assuming that  $d_0 = 1$  and  $d_{m+1} = 0$ ,  $E_k$  can be written as

$$E_k = (\alpha_1 - t) \sum_{i=0}^{k-1} d_i t^{k-i-1} - d_k.$$

As  $0 < t < \alpha_1 - t$  by the choice of  $t$  and  $|d_i| \leq \binom{m}{i} |\alpha_2|^i$  for  $1 \leq i \leq m$  by the definition of  $d_i$ ,

$$\begin{aligned} E_k &\geq t^k - \sum_{i=1}^k \binom{m}{i} |\alpha_2|^i (\alpha_1 - t)^{k-i} \\ &\geq (2t^m - (\alpha_1 - t + |\alpha_2|)^m) / t^{m-k}. \end{aligned}$$

By the assumption on  $\alpha_1$  and  $\alpha_2$ ,

$$2^{1/m} t \geq 2^{1/m} \left( \frac{\alpha_1}{2} - 1 \right) > \alpha_1 - \left( \frac{\alpha_1}{2} - 1 \right) + |\alpha_2| \geq \alpha_1 - t + |\alpha_2|$$

and we have  $E_k > 0$  for  $k = 1, \dots, m+1$ ; therefore by Lemma 2,

$$\sum_{i=1}^{m+1} e_i z^i / (1 - tz) \in \mathcal{F}. \quad (3.3)$$

Next, consider  $Q_N(z)z^{N+1}/(1 - tz)$ . By Lemma 3,  $q_N^i$  can be expressed as

$$q_N^i = q_N^0 d_i (1 + \epsilon_N^i),$$

where  $\lim_{N \rightarrow \infty} \epsilon_N^i = 0$  ( $i = 1, \dots, m$ ), and

$$\frac{Q_N(z) z^{N+1}}{1 - tz} = \frac{q_N^0 \{z + d_1(1 + \epsilon_N^1) z^2 + \dots + d_m(1 + \epsilon_N^m) z^{m+1}\} z^N}{1 - tz}.$$

For sufficiently large  $N$ , we can show that

$$t^k + d_1(1 + \epsilon_N^1)t^{k-1} + \dots + d_k(1 + \epsilon_N^k) > 0$$

by the same method as that used for showing  $E_k > 0$ , and as  $q_N^0 = a_{N+1}$  is a nonnegative integer, we have by Lemma 2

$$Q_N(z)z^{N+1}/(1 - tz) \in \mathcal{F}. \quad (3.4)$$

From (3.3), (3.4), and Lemma 1 we can conclude that

$$Q(N)z^{N+1}/P(z) \in \mathcal{F}$$

and this completes the proof.

Now consider the case where the inverse pole  $\alpha_1$  is multiple. The next lemma implies that the multiple case can be decomposed into simple cases.

LEMMA 5. Any polynomial with integral coefficients of the form  $P(z) = (1 - \alpha z)^K R(z)$ , where  $R(1/\alpha) \neq 0$  and  $K \geq 2$ , can be factored into  $(K + 1)$  polynomials with integral coefficients  $R_0(z), P_1(z), \dots, P_K(z)$ , that is,

$$P(z) = R_0(z) P_1(z) \cdots P_K(z),$$

where  $P_i(z) = (1 - \alpha z) R_i(z)$  ( $i = 1, \dots, K$ ) and  $R_i(1/\alpha) \neq 0$  ( $i = 0, 1, \dots, K$ ).

*Proof.* Consider the factorization of  $P(z)$  into irreducible polynomials with rational coefficients. As no irreducible equation on a field of characteristic 0 can have multiple roots, we know that  $P(z)$  can be factored into polynomials  $P_1(z), \dots, P_K(z)$  and  $R_0(z)$  with rational coefficients, where  $P_i(z) = (1 - \alpha z) R_i(z)$  and  $R_i(1/\alpha) \neq 0$ , and the coefficients of  $R_0(z), P_i(z)$  can be made integers by a theorem on polynomials (for example, Birkhoff and MacLane, 1969).

LEMMA 6. If  $f \in \mathcal{P}$  and  $\alpha_1 \gg |\alpha_2|, \dots, |\alpha_s|$  then  $f \in \mathcal{F}$ .

*Proof.* We expand  $f(z) = Q(z)/P(z) \in \mathcal{P}$  into a partial Taylor series and factor  $P(z)$  as in Lemma 5.

$$f(z) = \sum_{n=1}^N a_n z^n + f_N(z),$$

$$f_N(z) = \frac{Q_N(z)}{P_1(z) \cdots P_{K_1}(z) R_0(z)} z^{N+1}.$$

First we show that for sufficiently large  $N$

$$f_N(z) P_1(z) / z^N \in \mathcal{P}.$$

Let  $f_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$  and  $R_1(z) = (1 + d_1 z + \cdots + d_m z^m)$ ; then

$$\begin{aligned} f_N(z) P_1(z) &= \sum_{n=N+1}^{\infty} a_n z^n (1 - \alpha_1 z) (1 + d_1 z + \cdots + d_m z^m) \\ &= \sum_{n=N+1}^{\infty} g_n z^n, \end{aligned}$$

where

$$g_n = (a_n - \alpha_1 a_{n-1}) + d_1 (a_{n-1} - \alpha_1 a_{n-2}) + \cdots.$$

As  $a_n = q_{n+1}^0$  satisfies (3.2) for sufficiently large  $n$  and the characteristic equation of (3.2) has root  $\alpha_i$  of multiplicity  $K_i$ ,

$$a_n = \sum_{j=0}^{K_1-1} h_{1j} N^j \alpha_1^n + \sum_{j=2}^s \sum_{j=0}^{K_i-1} h_{ij} n^j \alpha_i^n. \quad (3.5)$$

As  $\alpha_1 \gg |\alpha_2|, \dots, |\alpha_s|$  and  $N$  is sufficiently large, putting  $K = K_1 - 1$ ,

$$g_n \simeq h_{1K} \alpha_1^n [n^K - (n-1)^K] > 0 \quad \text{for } n \geq N+1$$

(note that  $h_{1K} > 0$  as  $a_n \geq 0$  for all  $n \geq 1$ ) and  $f_N(z) P_1(z)/z^N$  satisfies condition (1) of Definition 3. Condition (2) is apparently satisfied by the assumption  $f \in \mathcal{P}$ , so

$$g(z) = f_N(z) P_1(z)/z^N \in \mathcal{P}.$$

As  $f_N(z) = [z^N/P_1(z)] g(z)$  and  $\alpha_1$  is an inverse pole of multiplicity  $K_1 - 1$  in  $g(z)$ , we can repeat the above process to obtain

$$\begin{aligned} f(z) &= \sum_{n=1}^{N_1} a_n z^n + \frac{z^{N_1}}{P_1(z)} \left\{ \sum_{n=1}^{N_2} b_n z^n + \frac{z^{N_2}}{P_2(z)} \right\} \cdots + \frac{z^{N_{K_1-1}}}{P_{K_1-1}(z)} \\ &\quad \times \left\{ \sum_{n=1}^{N_{K_1}} c_n z^n + \frac{h(z)}{P_{K_1}(z) R_0(z)} \right\} \cdots \left\{ \right\}, \end{aligned} \quad (3.6)$$

where  $a_n, b_n, \dots, c_n$  are nonnegative integers,  $N_1 = N$ , and  $h(z)/P_{K_1}(z) R_0(z) \in \mathcal{P}$ .

On the other hand, the assumption  $\alpha_1 \gg |\alpha_i|$  for  $i \geq 2$  implies  $z^{N_i}/P_i(z) \in \mathcal{P}$  as in the case of  $g(z)$  and as  $z^{N_i}/P_i(z)$  and  $h(z)/P_{K_1}(z) R_0(z)$  have the inverse pole  $\alpha_1$  of multiplicity 1 they belong to  $\mathcal{F}$  by Lemma 4; therefore  $f \in \mathcal{F}$ .

LEMMA 7. For a rational function  $f$  and a positive integer  $M$ , define  $[f]_M$  as

$$[f]_M = \sum_{j=0}^{M-1} f(\omega^j z)/M,$$

where  $\omega = \exp(2\pi\sqrt{-1}/M)$ ; then for each  $f \in \mathcal{P}$  there exist  $f_0, \dots, f_{M-1}$  such that the following hold.

- (1)  $f_i(z^M) = [z^{-i}f(z)]_M$ .
- (2)  $f_i(z) - f_i(0) \in \mathcal{P}$ .
- (3) If inverse poles of  $f$  are  $\alpha_1, \dots, \alpha_m$  then those of  $f_i$  are among  $\alpha_1^M, \dots, \alpha_m^M$ .
- (4)  $f(z) = \sum_{i=0}^{M-1} z^i f_i(z^M)$ .

*Proof.* Let

$$f(z) = \frac{Q(z)}{(1 - \alpha_1 z) \cdots (1 - \alpha_m z)} = \sum_{n=1}^{\infty} a_n z^n.$$

It can be easily shown that

$$\begin{aligned} [z^n]_M &= z^n && \text{when } n \text{ is a multiple of } M, \\ &= 0 && \text{otherwise,} \end{aligned}$$



and

$$[z^{-i}f(z)]_M = \sum_{n=0}^{\infty} a_{nM+i} z^{nM}$$

so  $[z^{-i}f(z)]_M$  can be expressed as a rational function whose denominator has no terms of the form  $z^k$  with  $k \not\equiv 0 \pmod{M}$ . Therefore, by considering the form of  $f(z)$ ,

$$[z^{-i}f(z)]_M = S(z)/R(z),$$

where

$$\begin{aligned} R(z) &= \prod_{j=0}^{M-1} (1 - \alpha_1 \omega^j z) \cdots (1 - \alpha_m \omega^j z) \\ &= (1 - \alpha_1^M z^M) \cdots (1 - \alpha_m^M z^M) \end{aligned}$$

and  $S(z)$  is a polynomial with (possibly) complex coefficients. Let  $r_1, \dots, r_m$  be

$$R(z) = 1 + r_1 z^M + \cdots + r_m z^{mM};$$

then  $r_i$  is a symmetric function of  $\alpha_1, \dots, \alpha_m$  and can be expressed as

$$r_i = A_i(c_1, \dots, c_m),$$

where  $A_i$  is an integral polynomial of  $m$ -variables and  $c_1, \dots, c_m$  are fundamental symmetric functions of  $\alpha_1, \dots, \alpha_m$ . As  $c_1, \dots, c_m$  are coefficients of the denominator  $(1 - \alpha_1 z) \cdots (1 - \alpha_m z)$  of  $f(z)$ ,  $r_i$  is an integer if  $f \in \mathcal{P}$ . As for  $S(z)$ ,

$$S(z) = R(z) \sum_{n=0}^{\infty} a_{nM+i} z^{nM}$$

implies that  $S(z)$  is also a polynomial of  $z^M$  with integral coefficients.

Now it is immediate that  $f_i(z)$  defined by

$$f_i(z) = \sum_{n=0}^{\infty} a_{nM+i} z^n$$

satisfies the properties stated and this completes the proof.

Now we state a procedure to show that  $\mathcal{P} \subset \mathcal{F}$ .

If  $f(z) \in \mathcal{P}$  then  $\alpha_1^{K_1} \alpha_2^{K_2} \cdots \alpha_s^{K_s} \geq 1$  and we have

(1)  $\alpha_1 > 1$  and for some  $p$ ,

$$\alpha_1 = |\alpha_2| = \cdots = |\alpha_p| > |\alpha_{p+1}| \geq \cdots \geq |\alpha_s|,$$

or

$$(2) \quad \alpha_1 = |\alpha_2| = \cdots = |\alpha_s| = 1. \quad (3.7)$$

First, consider case (1). Let  $r$  be the radius of convergence of  $f(z)$  and

$$\begin{aligned}\alpha_1 &= 1/r, \\ \alpha_i &= \exp(2\pi \sqrt{-1} k_i/l_i)/r \quad \text{for } 2 \leq i \leq p,\end{aligned}\tag{3.8}$$

where  $k_i, l_i$  : integers. For a sufficiently large common multiplier  $M$  of  $l_2, \dots, l_p$ , we represent  $f(z)$  as the sum of the  $z^i f_i(z^M)$ 's as in Lemma 7. Inverse poles of  $f_i(z) - f_i(0) \in \mathcal{P}$  are among  $1/r^M, \alpha_{p+1}^M, \dots, \alpha_s^M$ , where  $1/r^M \gg |\alpha_{p+1}^M|, \dots, |\alpha_s^M|$  and  $z^i f_i(z^M) = z^i [f_i(z^M) - f_i(0)] + f_i(0)z^i$ , so by Lemma 6,  $f(z) \in \mathcal{F}$  if every  $f_i(z)$  has the inverse pole  $1/r^M$ . When this condition does not hold and there exist  $f_j(z)$  whose inverse poles are  $\alpha_{q_1}^M, \dots, \alpha_{q_t}^M$ , where  $p+1 \leq q_1, \dots, q_t \leq s$ , we apply this procedure recursively to  $f_j(z) - f_j(0)$  as it is in  $\mathcal{P}$  and we can finally show that  $f(z) \in \mathcal{F}$  since  $t < s$ .

When condition (2) holds,  $f(z)$  can be expressed as the sum of the  $z^i f_i(z^M)$ 's for  $f_i(z) = h_i(z)/(1-z)^K$  as in case (1), where  $h_i(z)$  is a polynomial and  $M = \text{l.c.m.}(l_2, \dots, l_s)$ . By the method used in the proof of Lemma 6,  $f_i(z)$  can be transformed into

$$\begin{aligned}f_i(z) &= \sum_{n=0}^{N_1} a_n z^n + \frac{z^{N_1}}{1-z} \left\{ \sum_{n=1}^{N_2} b_n z^n + \frac{z^{N_1}}{1-z} \left\{ \dots + \frac{z^{N_{K-1}}}{1-z} \right. \right. \\ &\quad \times \left. \left. \left\{ \sum_{n=1}^{N_K} c_n z^n + \frac{dz^{N_{K+1}}}{1-z} \right\} \dots \right\} \right\},\end{aligned}$$

where  $a_n, b_n, \dots, c_n, d$  are nonnegative integers, and we can easily show that  $z^i f_i(z^M) \in \mathcal{F}$ , therefore  $f(z) \in \mathcal{F}$ .

From the above considerations we have the next theorem.

**THEOREM 2.**  $\mathcal{P} \subset \mathcal{F}$ .

The inverse inclusion is essentially due to Berstel (1971).

**THEOREM 3** (Berstel).  $\mathcal{P} \supset \mathcal{F}$ .

*Proof.* As  $z \in \mathcal{P}$  and  $\mathcal{P}$  is apparently closed under sum and product, we need only prove that if  $f \in \mathcal{P}$  then  $f^+ = f/(1-f) \in \mathcal{P}$ ; that is, every pole of  $[z^{-i}f(z)]_M$  on the circle of convergence is of the form  $r\epsilon$ , where  $r > 0$  and  $\epsilon$  is a root of unity. Let  $f(z) = Q(z)/P(z)$ , where  $P, Q$  are polynomials; then we can easily show that

$$[z^{-i}f^+(z)]_M = R(z^M)/(P(z^M) - Q(z^M))$$

for some polynomial  $R$ . Therefore every pole of  $[z^{-i}f^+(z)]_M$  is a root of  $f(z^M) = 1$  and if  $r$  is its radius of convergence then  $f(r^M) = 1$  by Pringsheim's theorem.

Let  $z_0 = r \exp(\sqrt{-1}\theta)$  be an arbitrary pole of  $[z^{-i}f(z)]_M$  on the circle of convergence; then

$$1 = |f(z_0^M)| \leq f(|z_0^M|) = f(r^M) = 1$$

and when  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $a_n \geq 0$ , we have

$$\left| \sum a_n r^{nM} \exp(\sqrt{-1} nM\theta) \right| = \sum a_n r^{nM},$$

so there exist integers  $p, q$  such that  $\theta = 2\pi p/q$  and this completes the proof.

Combining Theorems 1, 2, and 3 we have the next characterization theorem.

**THEOREM 4.** *A rational function  $f(z)$  with integral coefficients is a structure-generating function of an  $\epsilon$ -free regular set if and only if*

- (1) *the constant term of its denominator is 1 and that of its numerator is 0;*
- (2) *every coefficient  $a_n$  of the Taylor expansion in the neighborhood of  $z = 0$  is a nonnegative integer; and*
- (3) *every pole  $z_0$  of the minimal absolute value of  $f_i(z) = \sum_{n=0}^{\infty} a_{nM+i} z^n$  is of the form  $z_0 = r\epsilon$ , where  $r = |z_0|$  and  $\epsilon$  is a root of unity for any integer  $M \geq 1$  and  $i = 0, \dots, M-1$ .*

The structure-generating functions of regular sets are the same as the  $N$ -recognizable functions of Eilenberg (1974) and the above theorem also characterizes  $N$ -recognizable functions. By considering the details of our proof we see that it applies well also for  $R_+$ - and  $Q_+$ -recognizable functions. Here we define them as counterparts of  $\mathcal{F}$  (this is not the original definition of Eilenberg).

**DEFINITION 5.** A rational function with coefficients in  $R_+$  (or  $Q_+$ ) is  $R_+$ - (or  $Q_+$ -) recognizable iff it is obtained from  $z$  by repeated applications of multiplications by  $a \in R_+$  (or  $Q_+$ ), additions, products, and pseudo-inverses, where  $R_+$  (or  $Q_+$ ) is the set of nonnegative real (or rational) numbers.

**THEOREM 5.** *A rational function with coefficients in  $R_+$  (or  $Q_+$ ) is  $R_+$ - (or  $Q_+$ -) recognizable if and only if it satisfies:*

- (1) *The constant term of its denominator is nonzero and that of its numerator is zero;*
- (2) *every expansion coefficient  $a_n$  in the neighborhood of  $z = 0$  is in  $R_+$  (or  $Q_+$ ); and*
- (3) *every pole  $z_0$  of the minimal absolute value of  $f_i(z) = \sum_{n=0}^{\infty} a_{nM+i} z^n$  is of the form  $z_0 = r\epsilon$ , where  $r = |z_0|$  and  $\epsilon$  is a root of unity for any integer  $M \geq 1$  and  $i = 0, \dots, M-1$ .*

Finally, we give a theorem on the star height of regular sets.

DEFINITION 6. Regular sets  $R_1$  and  $R_2$  are *sgf*-equivalent iff  $g_{R_1} = g_{R_2}$ .

THEOREM 6. *For any regular set  $R_1$  there exists a regular set  $R_2$  which is sgf-equivalent to  $R_1$  and of star height at most 2.*

*Proof.* Let us consider our proof of  $\mathcal{P} \subset \mathcal{F}$ .

Suppose  $f(z) \in \mathcal{P}$  is given. First, repeated uses of Lemma 7 decompose  $f(z)$  into a sum of functions whose inverse poles  $\alpha_1, \alpha_2, \dots, \alpha_s$  satisfy  $\alpha_1 \gg |\alpha_2|, \dots, |\alpha_s|$  and  $\alpha_1 > 1$ , or  $\alpha_1 = \dots = \alpha_s = 1$  (hereafter we only consider the former case because subsequent processes are similar for the latter case). In this process no pseudo-inverse  $f^+$  is introduced. In the second step we may apply Lemma 6 to transform these functions into expressions such as the right-hand side of (3.6). Then Lemma 4 is used to process the functions  $z^{N_i}/P_i(z)$  and  $h(z)/P_{K_1}(z) R_0(z)$  and to transform them into expressions of the form

$$\sum_{n=1}^N a_n z^n + \frac{A(z)}{1-tz} \left/ \left( 1 - \frac{B(z)}{1-tz} \right) \right.,$$

where  $A(z), B(z)$  are polynomials, and functions of the form  $A(z)/(1-tz)$  are now expressed, by Lemma 2, as

$$A_1 z + \dots + A_{k-1} z^{k-1} + \frac{A_k z^k}{1-tz},$$

where the  $A_i$ 's are nonnegative integers, and shown to belong to  $\mathcal{F}$  using the pseudo-inverse only once. Therefore the depth of the nest of pseudo-inverses in the construction of  $f$  according to the syntax of  $\mathcal{F}$  is at most 2 and this completes the proof.

#### 4. CHARACTERIZATION OF DOL GROWTH FUNCTIONS

Our method is also applicable to the characterization of the class of generating functions of the growth functions of *DOL* systems. In this case the situation is somewhat complicated by the fact that functions of the form  $[f]_M$ ,  $M \geq 2$ , cannot be the generating functions of the *DOL* growth functions and we must use Lemma 7 before Lemmas 4 and 6.

In the following we state characterization theorems of *DOL* growth functions but only the outlines of the proof are stated because the techniques used are very similar to those for the case of regular sets and are easily inferred from them.

DEFINITION 7. A *DOL* system is a triple  $G = (A, \delta, \omega)$ , where  $A$  is a finite set of letters,  $\delta$  is a homomorphism from  $A$  into  $A^*$ , and  $\omega \in A^*$ . The growth function  $a_G(n)$  of  $G$  is defined by  $a_G(n) = \lg(\delta^n(\omega))$ , where  $\lg$  denotes the length of the word and  $\delta^n$  is the  $n$ -fold composition of  $\delta$ . The generating function  $f_G(z)$  of  $a_G(n)$  is given by

$$f_G(z) = \sum_{n=0}^{\infty} a_G(n) z^n$$

and we denote the class of all such generating functions by  $\mathcal{D}$ , that is,

$$\mathcal{D} = \{f_G(z) \mid G \text{ is a DOL system}\}.$$

DEFINITION 9.  $\mathcal{C}$  is a class of rational functions of the form

$$f(z) = \frac{q_0 + q_1 z + \cdots + q_m z^m}{1 + p_1 z + \cdots + p_l z^l} = \sum_{n=0}^{\infty} a_n z^n,$$

where the  $p_i$ 's and  $q_i$ 's are integers, and satisfy either:

- (1) there exists  $N$  such that  $a_n > 0$  for  $n < N$  and  $a_n = 0$  for all  $n \geq N$ ,  
or
- (2)  $a_n > 0$  and  $a_{n+1}/a_n$  is bounded for any  $n$ , and every pole  $z_0$  of the minimal absolute value is of the form  $z_0 = r\epsilon$ , where  $r = |z_0|$  and  $\epsilon$  is a root of unity.

*Notation.* In the following we assume that symbols and terms have the same meanings as in the previous section if they are not explicitly defined.

The purpose of this section is to show that  $\mathcal{D} = \mathcal{C}$ .  $\mathcal{D} \subset \mathcal{C}$  is immediate from Berstel's theorem and the definition of *DOL* systems. Every polynomial in  $\mathcal{C}$  is apparently in  $\mathcal{D}$ ; therefore we need only show that  $\mathcal{D} \supset \mathcal{C}_0$ , where  $\mathcal{C}_0$  is a subset of  $\mathcal{C}$  such that every function in  $\mathcal{C}_0$  is a nonpolynomial rational function.

We assume every  $f(z) = Q(z)/P(z) \in \mathcal{C}_0$  is of form (3.1) with the same constraints (i)  $\alpha_1 > 0$ , (ii)  $\alpha_1 \geq |\alpha_2| \geq \cdots \geq |\alpha_s|$ , and (iii)  $Q(1/\alpha_1) \neq 0$ .

First we give a closure property of class  $\mathcal{D}$ .

DEFINITION 10. For rational functions  $f(z)$  and  $g(z)$ , (1) *shift* and (2) *quasi-quotient* of  $f, g$  are defined by

- (1)  $\sum_{n=0}^N a_n z^n + z^{N+1} g(z)$ , where  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  
and
  - (2)  $g(z)/(1 - f(z^M)z^M)$ , where  $M$  is a positive integer,
- respectively.

LEMMA 8.  $\mathcal{D}$  is closed under (1) addition, (2) shift, and (3) quasi-quotient.

*Proof.* Closure under addition and shift is due to Ruohonen (1975). Closure under quasi-quotient can be proved in the identical way as Theorem 6 of Ruohonen (1975) after one notes that  $\mathcal{D}$  is closed under quasi-inverse (a quasi-inverse of  $f(z)$  is defined by  $1/(1 - zf(z))$ ).

In the following we give three lemmas, which correspond to Lemmas 7, 4, and 6, respectively, in the case of regular sets.

LEMMA 9. *For a rational function  $f \in \mathcal{C}_0$  and a positive integer  $M$ , define  $[f]_M$  as in Lemma 7; then there exist  $f_0, \dots, f_{M-1} \in \mathcal{C}_0$  such that*

$$(1) \quad f_i(z^M) = [z^{-i}f(z)]_M,$$

(2) *if inverse poles of  $f$  are  $\alpha_1, \dots, \alpha_s$  then those of  $f_i(z)$  are among  $\alpha_1^M, \dots, \alpha_s^M$ . Furthermore multiplicities of  $\alpha_i^M$  in the  $f_i$ 's are equal.*

$$(3) \quad f(z) = \sum_{i=0}^{M-1} z^i f_i(z^M).$$

*Proof.* Conditions (1) and (3) and the first half of (2) are proved as in Lemma 7. As for the latter half of (2), suppose there exist  $i$  and  $j$  such that the multiplicity of  $\alpha_1^M$  in  $f_i(z)$  is larger than that in  $f_j(z)$ .

As expansion coefficients  $a_{i,n}$  and  $a_{j,n}$  of  $f_i$  and  $f_j$  are given by expressions such as the right-hand side of (3.5) and  $a_{nM+k} = a_{k,n}$  for  $k \leq M-1$ , we can easily show that  $a_{nM+i}/a_{nM+j} = a_{i,n}/a_{j,n}$  is not bounded and this contradicts the boundedness of  $a_{n+1}/a_n$ . Therefore the latter half of (2) holds.

LEMMA 10. *Suppose  $f_0(z), \dots, f_{M-1}(z) \in \mathcal{C}_0$  have a common denominator, that is,  $f_i(z) = Q_i(z)/P(z)$  for  $i = 0, \dots, M-1$ . If the multiplicity  $K_1$  of the inverse pole  $\alpha_1$  is 1 and  $\alpha_1 \gg |\alpha_2|, \dots, |\alpha_s|$ , then  $f(z) = \sum_{i=0}^{M-1} z^i f_i(z^M) \in \mathcal{D}$ .*

*Proof.* For sufficiently large  $N$ , we expand  $f_i$  into a partial Taylor series as in (1.1):

$$f_i(z) = \sum_{n=0}^N a_{i,n} z^n + \frac{Q_{iN}(z)}{P(z)} z^{N+1}.$$

Then

$$f(z) = \sum_{i=0}^{M-1} \sum_{n=0}^N a_{i,n} z^{nM+i} + \frac{\sum_{i=0}^{M-1} Q_{iN}(z^M) z^i}{P(z^M)} z^{(N+1)M}.$$

As  $a_{i,n} > 0$  for all  $i, n$  by  $f_i \in \mathcal{C}_0$  we have

$$\sum_{i=0}^{M-1} \sum_{n=0}^N a_{i,n} z^{nM+i} \in \mathcal{D}$$

and we need only prove that  $g(z) = \sum_{i=0}^{M-1} Q_{iN}(z^M) z^i / P(z^M) \in \mathcal{D}$ . We transform  $g(z)$  as in the proof of Lemma 4:

$$g(z) = \frac{\sum_{i=0}^{M-1} Q_{iN}(z^M) z^i}{1 - tz^M} \left/ \left( 1 - \frac{\sum_{i=1}^{m+1} e_i z^{(i-1)M}}{1 - tz^M} z^M \right) \right.$$

As each  $Q_{iN}(z)/(1 - tz)$  can be transformed into the form

$$\frac{Q_{iN}(z)}{1 - tz} = b_{i,0} + b_{i,1}z + \cdots + b_{i,m-1}z^{m-1} + \frac{b_{i,m}z^m}{1 - tz},$$

where  $b_{i,j} > 0$  by the choice of  $t$ , we have

$$\frac{\sum_{i=0}^{M-1} Q_{iN}(z^M) z^i}{1 - tz^M} = \sum_{j=0}^{m-1} \sum_{i=0}^{M-1} b_{i,j} z^{jM+i} + \frac{\sum_{i=0}^{M-1} b_{i,m} z^i}{1 - tz^M} z^{mM} \in \mathcal{D}, \quad (4.1)$$

as  $\mathcal{D}$  is closed under shift operation and  $\sum_{i=0}^{M-1} b_{i,m} z^i / (1 - tz^M) \in \mathcal{D}$  by Lemma 8 (take  $f_1 = \sum b_{i,m} z^i$ ,  $f_2 = t$ ).

On the other hand we can easily verify that

$$\sum_{i=1}^{m+1} e_i z^{i-1} / (1 - tz) \in \mathcal{D} \quad (4.2)$$

from the argument used in showing (3.3). Therefore by (4.1), (4.2), and Lemma 8 we can conclude that  $g(z) \in \mathcal{D}$  and this completes the proof.

LEMMA 11. If  $f_0(z), \dots, f_{M-1}(z) \in \mathcal{C}_0$  have a common denominator, that is,  $f_i(z) = Q_i(z)/P(z)$  for  $i = 0, \dots, M-1$ , and  $\alpha_1 \gg |\alpha_2|, \dots, |\alpha_s|$  then

$$f(z) = \sum_{i=0}^{M-1} z^i f_i(z^M) \in \mathcal{D}.$$

*Proof.* Every  $f_i(z)$  can be transformed into a form such as (3.6) in the proof of Lemma 6,

$$\begin{aligned} f_i(z) &= \sum_{n=0}^{N_i} a_{i,n} z^n + \frac{z^{N_1+1}}{P_1(z)} \left\{ \sum_{n=0}^{N_2} b_{i,n} z^n + \frac{z^{N_2+1}}{P_2(z)} \left\{ \cdots + \frac{z^{N_{K_1-1}+1}}{P_{K_1-1}(z)} \right. \right. \\ &\quad \times \left. \left. \left\{ \sum_{n=0}^{N_{K_1}} c_{i,n} z^n + \frac{h_i(z)}{P_{K_1}(z) R_0(z)} \right\} \cdots \right\} \right\}, \end{aligned}$$

where (i)  $a_{i,n}, b_{i,n}, \dots, c_{i,n}$  are positive integers, (ii)  $P_i(z) = (1 - \alpha_1 z) R_i(z)$ ,  $R_i(1/\alpha_1) \neq 0$  for  $i = 1, \dots, K_1$  and (iii)  $h_i(z)/P_{K_1}(z) R_0(z) z^{N_{K_1}+1} \in \mathcal{C}_0$ ,  $R_0(1/\alpha_1) \neq 0$ , so

$$\begin{aligned} f(z) &= \sum_{n=0}^{N_1} \sum_{i=0}^{M-1} a_{i,n} z^{nM+i} + \frac{z^{(N_1+1)M}}{P_1(z^M)} \left\{ \sum_{n=0}^{N_2} \sum_{i=0}^{M-1} b_{i,n} z^{nM+i} + \frac{z^{(N_2+1)M}}{P_2(z^M)} \right. \\ &\quad \times \left\{ \cdots + \frac{z^{(N_{K_1-1}+1)M}}{P_{K_1-1}(z^M)} \left\{ \sum_{n=0}^{N_{K_1}} \sum_{i=0}^{M-1} c_{i,n} z^{nM+i} + \sum_{i=0}^{M-1} \frac{h_i(z^M)}{P_{K_1}(z^M) R_0(z^M)} z^i \right\} \cdots \right\} \right\}. \end{aligned} \quad (4.3)$$

By Lemma 10 we immediately have

$$\sum_{i=0}^{M-1} \frac{h_i(z^M)}{P_{K_1}(z^M) R_0(z^M)} z^i \in \mathcal{D}. \quad (4.4)$$

On the other hand, for any  $d(z) \in \mathcal{D}$ ,  $d(z)/P_k(z^M)$  can be transformed, as in the proof of Lemma 4, into

$$\frac{d(z)}{P_k(z^M)} = \frac{d(z)}{1 - tz^M} \left/ \left( 1 - \frac{p(z^M)}{1 - tz^M} z^M \right) \right., \quad (4.5)$$

where  $p(z)$  is a polynomial such that  $p(z)/(1 - tz) \in \mathcal{D}$ , and can be shown to belong to  $\mathcal{D}$  by using Lemma 8 twice.

Therefore, from (4.3), (4.4), and repeated use of the fact that  $\mathcal{D}$  is closed under the shift operation we have  $f(z) \in \mathcal{D}$  and this completes the proof.

From the lemmas hitherto stated we can easily derived  $\mathcal{C}_0 \subset \mathcal{D}$ . Here we describe a procedure to show that a given  $f(z) \in \mathcal{C}_0$  belongs to  $\mathcal{D}$ .

Inverse poles of  $f(z)$  satisfy condition (1) or condition (2) of (3.7) and we only consider case (1) since case (2) can be treated in a similar way.

First we decompose  $f(z)$  into the sum of  $z^i f_i(z^M)$ 's as in Lemma 9 for a sufficiently large common multiplier  $M$  of  $l_2, \dots, l_p$ , which are defined in (3.8). As inverse poles of every  $f_i(z)$  are  $\alpha_1^M, \alpha_{p+1}^M, \dots, \alpha_s^M$ , where  $\alpha_1^M \gg |\alpha_{p+1}^M|, \dots, |\alpha_s^M|$  and all  $f_i(z)$  can be made to have a common denominator, that is,  $f_i(z) = Q_i(z)/P(z)$  such that  $Q_i(1/\alpha_1^M) \neq 0$ , by the latter half of (2) of Lemma 9. Therefore  $f(z) \in \mathcal{D}$  by Lemma 11 and we have  $\mathcal{C}_0 \subset \mathcal{D}$ .

**THEOREM 7.** *A rational function  $f(z)$  with integral coefficients is a generating function for a DOL growth function if and only if (i)  $f(z)$  is a polynomial  $a_0 + a_1 z + \dots + a_N z^N$  with  $a_0, \dots, a_N > 0$ , or (ii)  $f(z)$  satisfies:*

- (1) *The constant term of its denominator is 1;*
- (2) *its coefficients  $a_n$ 's of the Taylor expansion in the neighborhood of  $z = 0$  satisfy  $a_n > 0$  and  $a_{n+1}/a_n$  : bounded for all  $n \geq 0$ ; and*
- (3) *every pole  $z_0$  of the minimal absolute value is of the form  $z_0 = r\epsilon$ , where  $r = |\alpha_0|$  and  $\epsilon$  is a root of unity.*

The next theorem is a corollary of Theorem 7 and characterizes generating functions for a DOL growth function in terms of functional composition and can be easily obtained by examining the right-hand sides of (4.3) and (4.5) and Lemma 8.

**THEOREM 8.** *The class of generating functions for DOL growth functions is a minimal class of rational functions such that*



- (i) *it contains the constant 1; and*
- (ii) *it is closed under addition, shift, and quasi-quotient.*

*Furthermore, in constructing any function in this class using the above operations, the depth of the nest of the quasi-quotient can be made at most 2.*

#### ACKNOWLEDGMENTS

The authors wish to thank Professor Samuel Eilenberg for constructive comments, Professor Arto Salomaa for informing us of Soittola's results, and Professor Jean Berstel for pointing out errors in the original versions of Definition 3 and Lemma 7. We also wish to thank the referees for their suggestions for improving the paper.

RECEIVED: December 31, 1975; REVISED: January 27, 1977

#### REFERENCES

- BERSTEL, J. (1971), Sur les poles et le quotient de Hadamard de series  $N$ -rationnelles, *C. R. Acad. Sci. Paris* **272**, 1079–1081.
- BIRKHOFF, G., AND MACLANE, S. (1969), "A Survey of Modern Algebra," Macmillan, New York.
- EILENBERG, S. (1974), "Automata, Languages and Machines," Vol. A, Academic Press, New York.
- KUICH, W. (1970), On the entropy of context-free languages, *Inform. Contr.* **16**, 173–200.
- RUOHONEN, K. (1975), On the synthesis of DOL Growth, *Ann. Acad. Sci. Fenn. Ser. A. I.* **1**, 143–154.
- SOITTOLA, M. (1976a), Positive rational sequences, *Theoret. Comput. Sci.* **2**, 317–322.
- SOITTOLA, M. (1976b), Remarks on DOL growth sequences, *Rev. Française Automat. Informat. Recherch Opérationnelle* **10**, 23–24.